UDC 539.3

AXISYMMETRIC CONTACT PROBLEM OF THE THEORY OF ELASTICITY IN THE PRESENCE OF WEAR

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A solution is given for the contact problem of the theory of elasticity, in which a stamp bounded by a surface of revolution is impressed into an elastic half-space, with the wear of the half-space caused by rotation of the stamp taken into account. It is assumed that the wear is abrasive [1, 2] and, that the stamp is not displaced along its axis when the wear takes place. This implies that the pressure between the stamp and the foundation will diminish with time.

A stamp with the end surface defined by the equation $z = f(\rho)$ is situated on an elastic half-space, and is acted upon by the force P acting along the axis of rotation and a moment M rotating it about this axis. The force and the moment both vary with time (see Fig. 1.). The angular velocity of rotation is constant and equal to ω. The plane of contact is a circle of radius $a \leqslant b$ where b is the radius of the cylindrical part of the stamp surface. When the stamp rotates, frictional forces $\tau_{z\theta}$ appear at surface of contact, acting in the direction of rotation, i. e. they are perpendicular to the radius of the area of contact, and

$$\tau_{z\theta} = \mu\sigma_z$$

where σ_z is the normal pressure at the plane of contact and μ is the coefficient of friction between the bodies in contact. It must be remembered that in the case considered the stresses σ_z, σ_ρ, σ_θ, $\tau_{\rho z}$, $\tau_{z\theta}$, $\tau_{\rho\theta}$ and displacements u_z, u_ρ and u_θ are all functions of time t.

The following boundary conditions at the area of contact, at $z = 0$ and the initial instant of time, are used to determine the stressed state (S is a circle of radius a, and the condition outside is formulated at the free surface):

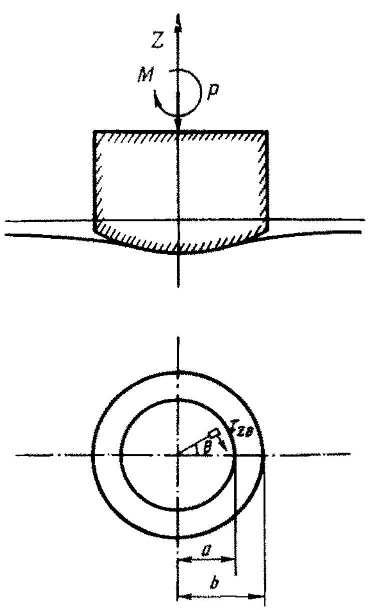

Fig. 1

$$u_z = f(\rho), \quad \tau_{z\theta} = \mu\sigma_z, \quad \tau_{\rho z} = 0 \text{ on } S \tag{1}$$
$$\sigma_z = 0, \quad \tau_{z\theta} = 0, \quad \tau_{\rho z} = 0 \quad \text{outside } S$$

The displacement of the foundation $w(\rho, t)$ in the direction of the z-axis at any instant of time is represented by the difference between the initial displacement $w_0(\rho) = f(\rho)$ and the displacement of the foundation due to wear. For this reason the stressed state of the elastic half-space at any instant of time is determined by the following boundary conditions:

$$u_z = w(\rho, t), \quad \tau_{z\theta} = \mu\sigma_z, \quad \tau_{\rho z} = 0 \text{ on } S \tag{2}$$

$$\sigma_z = 0, \quad \tau_{z\theta} = 0, \quad \tau_{\rho z} = 0 \text{ outside } S$$

We solve the problem using the Lamé equations written in cylindrical coordinates with the axial symmetry of the problem taken into account, as follows:

$$(\lambda + 2\mu)\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial(\rho u_\rho)}{\partial\rho} + \frac{\partial u_z}{\partial z}\right] + \mu\frac{\partial}{\partial z}\left[\frac{\partial u_\rho}{\partial z} - \frac{\partial u_z}{\partial\rho}\right] = 0 \tag{3}$$
$$(\lambda + 2\mu)\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial(\rho u_\rho)}{\partial\rho} + \frac{\partial u_z}{\partial z}\right] - \frac{\mu}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial u_\rho}{\partial z} - \rho\frac{\partial u_z}{\partial\rho}\right] = 0$$
$$\mu\frac{\partial}{\partial z}\left(\frac{\partial u_\theta}{\partial z}\right) + \mu\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial(\rho u_\theta)}{\partial\rho}\right] = 0$$

Taking into account the relations connecting the stress and displacement components we see that in the axisymmetric case the elastic body has two independent systems of deformations and stresses. The first system is obtained by setting $u_\theta = 0$, $u_\rho \neq 0$, $u_z \neq 0$. Here the stress components $\tau_{\rho\theta}$ and $\tau_{z\theta}$ are zero and the remaining ones are not zero. The second system can be obtained by setting $u_\rho = u_z = 0$, $u_\theta \neq 0$, in which case the stress components are $\sigma_\theta = \sigma_\rho = \tau_{\rho z} = 0$ and $\tau_{\rho\theta} \neq 0$, $\tau_{z\theta} \neq 0$.

The state of stress in the problem of impressing a stamp of circular cross-section rotating about its own axis into an elastic half-space, can be separated into two independent stress states. We determine the first state using the following boundary conditions at the plane of contact $z = 0$:

$$u_z^* = w(\rho, t), \quad \tau_{z\theta}^* = 0, \quad \tau_{\rho z}^* = 0 \text{ on } S \tag{4}$$
$$\sigma_z^* = 0, \quad \tau_{z\theta}^* = 0, \quad \tau_{\rho z}^* = 0 \text{ outside } S$$

The second state of stress is obtained from the conditions at the boundary $z = 0$:

$$u_z^{**} = 0, \quad \tau_{z\theta}^{**} = \mu\sigma_z^*, \quad \tau_{\rho z}^{**} = 0 \text{ on } S \tag{5}$$
$$\sigma_z^{**} = 0, \quad \tau_{z\theta}^{**} = 0, \quad \tau_{\rho z}^{**} = 0 \text{ outside } S$$

The sum of these two states of stress (i.e. the stresses $\sigma_z = \sigma_z^* + \sigma_z^{**}$, $\tau_{z\theta} = \tau_{z\theta}^* + \tau_{z\theta}^{**}$ etc.) will satisfy the boundary conditions (2), i.e. will represent a solution of the problem in question.

We see from (4) that the quantities σ_z^*, $\tau_{z\theta}^*$ etc., can be found from the solution of the problem of impressing a stamp representing a solid revolution, into an elastic half-space without friction. The displacements u_z^* at $z = 0$ are connected in this problem with the normal stresses σ_z^* $(z = 0)$ by the relation [3] which has the following form in polar coordinates in the case of axial symmetry:

$$u_z^*(\rho, t) = -\frac{1-\nu^2}{\pi E}\int_0^a\int_0^{2\pi}\frac{\sigma_z^*(r, t)\, r\, dr\, d\varphi}{\sqrt{r^2+\rho^2-2r\rho\cos\varphi}} \tag{6}$$

From a solution of the problem satisfying the boundary conditions (5) it follows that $u_z^{**} = 0$ and $\sigma_z^{**} = 0$ on the circle, as well as outside of the circle S. At $z = 0$ the shear stress $\tau_{z\theta}^{**}$ can be found, provided that a solution of the boundary value problem (4) for the normal stress σ_z^* acting at the plane of contact, is known. Clearly, $\tau_{z\theta}^{**} = \tau_{z\theta}$, $\sigma_z^* = \sigma_z = -p(\rho, t)$ and $u_z^* = u_z = w(\rho, t)$. Thus the displacements $w(\rho, t)$ taking place at the plane of contact are connected with the normal pressure $p(\rho, t)$ acting at the boundary of the elastic half-space by the following relation which holds at any instant of time:

$$w(\rho, t) = \frac{1-\nu^2}{\pi E}\int_0^a\int_0^{2\pi}\frac{p(r, t)\, r\, dr\, d\varphi}{\sqrt{r^2+\rho^2-2r\rho\cos\varphi}} \tag{7}$$

The shear stresses at the boundary of the elastic half-space are defined as follows:

$$\tau_{z\theta}(\rho, t) = -\mu p(\rho, t) \tag{8}$$

Here it must be remembered that the radius of the area of contact changes as the result or wear. However, since the wear is small compared with the dimension of the area of contact, the above aspect can be neglected.

Let us write the equation (7) in dimensionless coordinates

$$w_1(\rho_1, t) = \frac{1-\nu^2}{\pi E}\int_0^1\int_0^{2\pi}\frac{p_1(r_1, t)\, r_1\, dr_1\, d\varphi}{\sqrt{r_1^2+\rho_1^2-2r_1\rho_1\cos\varphi}} \tag{9}$$

$$\rho_1 = \rho / a, \quad r_1 = r / a, \quad w_1(\rho_1, t) = w(\rho_1 a, t) / a$$

$$p_1(\rho_1, t) = p(\rho_1 a, t)$$

We find the displacement $w(\rho, t)$ of the boundary of the elastic half-space which will vary as the result of wear. When the wear is abrasive, the amount of the material removed can be assumed proportional to the work done by the frictional forces [1, 2]. In this case, taking (8) into account, we can define the modulus of the rate at which the displacement of the points of the boundary of the half-plane varies, as follows:

$$|\partial w(\rho, t) / \partial t| = k\omega\rho\, |\tau_{z\theta}| = k\omega\rho\mu p(\rho, t) \tag{10}$$

where k is the proportionality coefficient connecting the work done by the frictional forces and the amount of material removed. In this case we obtain the following expression (in dimensionless coordinates) for the displacements of the points of the area of contact:

$$w_1(\rho_1, t) = w_{01}(\rho_1) - k\omega\rho_1\mu \int_0^t p_1(\rho_1, \tau)\, d\tau \tag{11}$$

$$w_{01}(\rho_1) = w(\rho_1 a, 0) / a$$

If the proportionality and friction coefficients k and μ are constant, then the displacement due to wear will be zero at the center of the area of contact and this should lead to increased stresses at this point. This in turn will cause irreversible plastic deformations at the center of the area of contact. Thus although irreversible changes of form occur over the whole area of contact, the solution of the problem of the theory of elasticity given below will be valid for the whole zone of contact except a small region near its center.

Let us introduce the function $q(\rho_1, t)$ connected with the pressure $p_1(\rho_1, t)$ in the following manner:

$$p_1(\rho_1, t) = q(\rho_1, t) / \rho_1 \tag{12}$$

Then the following equation for the function $q(\rho_1, t)$ at any instant of time follows from (9) and (11)

$$\frac{1-\nu^2}{\pi E}\int_0^1\int_0^{2\pi} \frac{q(r_1, t)\, dr_1\, d\varphi}{\sqrt{r_1^2+\rho_1^2-2r_1\rho_1\cos\varphi}} = w_{01}(\rho_1) - k\omega\mu\int_0^t q(\rho_1, \tau)\, d\tau \tag{13}$$

Here $w_{01}(\rho_1)$ represents the equation of the end surface of the stamp written in dimensionless coordinates. The initial displacement has the corresponding distribution of pressure $p_1(\rho_1, 0) = q(\rho_1, 0) / \rho_1$, which can be found from the solution of the boundary value problem (1) and is related to the displacement $w_{01}(\rho_1)$ as follows:

$$w_{01}(\rho_1) = \frac{1-\nu^2}{\pi E}\int_0^1\int_0^{2\pi} \frac{q(r_1, 0)\, dr_1\, d\varphi}{\sqrt{r_1^2+\rho_1^2-2r_1\rho_1\cos\varphi}} \tag{14}$$

We shall seek the particular solutions of (13) in the form

$$q(\rho_1, t) = q_\beta(\rho_1)e^{-\beta t}$$

Then the quantities $q_\beta(\rho_1)$ will be given, with (14) taken into account, by the homogeneous Fredholm equation with a symmetric kernel

$$q_\beta(\rho_1) - \beta\int_0^1 H(r_1, \rho_1)\, q_\beta(r_1)\, dr_1 = 0 \tag{15}$$

$$H(r_1, \rho_1) = \frac{1-\nu^2}{k\omega\mu\pi E}\int_0^{2\pi} \frac{d\varphi}{\sqrt{r_1^2+\rho_1^2-2r_1\rho_1\cos\varphi}}$$

The kernel $H(r_1, \rho_1)$ can be written using a complete elliptic integral of the first kind, in the form

$$H(r_1, \rho_1) = \frac{4(1-\nu^2)}{k\omega\mu\pi E(r_1+\rho_1)} \mathbf{K}\left(\frac{2\sqrt{r_1\rho_1}}{r_1+\rho_1}\right)$$

Equation (15) enables us to determine the eigenvalues β_n, which will all be real since the kernel is symmetric and real. We shall also show that all eigenvalues β_n are positive. This will be true if and only if the kernel $H(r_1, \rho_1)$ is positive definite, i.e.

$$J(q) = \int_0^1\int_0^1 H(r_1, \rho_1)\, q(r_1)\, q(\rho_1)\, dr_1\, d\rho_1 > 0$$

for every continuous function $q(r_1)$ not identically equal to zero in the interval $(0, 1)$. The functional $J(q)$ can be written in the form (making use of the structure of the formulas (14) and (12))

$$J(q) = \int_0^1 q(\rho_1)\left\{\frac{1-\nu^2}{k\omega\mu\pi E}\int_0^1\int_0^{2\pi}\frac{q(r_1)\, dr_1\, d\varphi}{\sqrt{r_1^2+\rho_1^2-2r_1\rho_1\cos\varphi}}\right\} d\rho_1 =$$

$$\frac{1}{k\omega\mu}\int_0^1 p_1(\rho_1, 0)\, w_{01}(\rho_1)\, \rho_1\, d\rho_1$$

Thus the functional $J(q)$ represents, with the accuracy of up to the positive multiplier, the total work done by the arbitrary forces of pressure $p_1(\rho_1, 0) = q(\rho_1)$ on the corresponding displacements of the points of the contact area at the initial instant of time. When the pressures are not zero, the work is always positive.

The eigenfunctions of the integral equation (15) will be orthogonal by virtue of the symmetry of the kernel. The initial pressure $p_1(\rho_1, 0)$ is given by the formula [3]

$$p_1(\rho_1, 0) = -\frac{E}{4\pi(1-\nu^2)}\int_0^1 \Delta w_{01}(r_1)\, L(r_1, \rho_1)\, dr_1 \tag{16}$$

$$L(r_1, \rho_1) = \int_0^{2\pi}\frac{2r_1}{\pi\sqrt{r_1^2+\rho_1^2-2r_1\rho_1\cos\varphi}} \times$$

$$\operatorname{arctg}\frac{\sqrt{1-r_1^2}\sqrt{1-\rho_1^2}}{\sqrt{r_1^2+\rho_1^2-2r_1\rho_1\cos\varphi}}\, d\varphi$$

$$\Delta = \left(\frac{1}{r_1}\frac{\partial}{\partial r_1} + \frac{\partial^2}{\partial r_1^2}\right)$$

Expanding the function $q(\rho_1, 0) = \rho_1 p_1(\rho_1, 0)$, where $p_1(\rho_1, 0)$ is given by (16), into a series in terms of the complete orthonormal set of eigenfunctions $q_n(\rho_1)$ of the integral equation (15), we obtain the coefficients A_n

$$q(\rho_1, 0) = \sum_{n=1}^{\infty} A_n q_n(\rho_1)$$

Then the pressure at the subsequent instants of time will be given by the formula

$$p_1(\rho_1, t) = \frac{q(\rho_1, t)}{\rho_1} = \frac{1}{\rho_1} \sum_{n=1}^{\infty} A_n q_n(\rho_1) e^{-\beta_n t}$$

It is clear that the expression for the pressure in the axisymmetric contact problem with wear has the same form as that in the two-dimensional contact problem of motion of a stamp along the boundary of an elastic layer [4].

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